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LETTER TO THE EDITOR

Quantum Borel kinematics on three-dimensional manifolds and knot groups

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Abstract. The quantizations of n identical or distinguishable particles, which are localized on closed orientable 3-manifolds, viewed as a three-fold branched covering of S^3 with a matching knot chosen as branching set, are classified up to unitary equivalence. The result connects the set η of non-equivalent quantizations to knot theory.

1. Introduction

Consider a system of n identical or distinguishable non-relativistic particles without spin, which are localized on a differentiable manifold M. If the particles are distinguishable the configuration manifold F_n is given by $F_n = (M \times \cdots \times M) - D$, if the particles are identical the effective configuration manifold is $\Delta_n = ((M \times \cdots \times M) - D)/S_n$. D denotes the diagonal, i.e. the set of points describing configurations where at least two particles coincide. Δ_n and F_n are, in general, topologically non-trivial.

To quantize the system on Δ, i.e. with $\Delta = \Delta_n$ or $\Delta = F_n$, a quantization method is necessary, which incorporates the global structure of F_n and Δ_n. The Borel quantization [2, 8] is such a method. It is based on the generalized position $\boldsymbol{Q}(f)$ and momentum operators $\boldsymbol{P}(X)$ modelled via functions $f \in C^\infty(\Delta, \mathbb{R})$ and smooth (complete) vector fields $X \in \mathcal{X}_c(\Delta)$, respectively, for systems on arbitrary smooth manifolds Δ with measure μ. The functions and the vector fields have a natural Lie structure and span the covariance algebra $S(\Delta)$:

$$S(\Delta) = C^\infty(\Delta, \mathbb{R}) \ni\!\!\!\!\!\!+ \mathcal{X}_c(\Delta).$$

The quantization is constructed via a quantization map $(\boldsymbol{Q}, \boldsymbol{P})$ of $S(\Delta)$ into the set of self-adjoint operators on the Hilbert space $\mathcal{H} = L^2(\Delta, \mu)$,

$$\boldsymbol{Q} : C^\infty(\Delta, \mathbb{R}) \ni f \longrightarrow \boldsymbol{Q}(f) \in \mathcal{L}_s(\mathcal{H})$$
$$\boldsymbol{P} : \mathcal{X}_c(\Delta) \ni X \longrightarrow \boldsymbol{P}(X) \in \mathcal{L}_s(\mathcal{H})$$

such that: (1) the Lie structure is conserved; (2) $\boldsymbol{Q}(f)$ is a multiplication operator; (3) to have differential operators for $\boldsymbol{P}(X)$ one needs differential structures $\mathcal{D}$ on the point set $\Delta \times \mathbb{C}$. Suitable $\mathcal{D}$ are induced by a diffeomorphic (isomorphic on each fibre) mapping from $\Delta \times \mathbb{C}$ onto a line bundle $L = (E, \mathrm{pr}, \Delta, \mathbb{C})$ with total space E and projection pr. The Hilbert space $L^2(\Delta, \mu)$ of functions is viewed as the Hilbert space $L^2(L, \mu)$ of sections, i.e. as the completion of the space of square integrable sections in the Hermitean linebundle $(L, \langle\cdot, \cdot\rangle)$ (4). $\boldsymbol{P}(X)$ is local in the sense that $\boldsymbol{P}(X)\Psi = 0$ if $\mathrm{supp}(X) \cap \mathrm{supp}(\Psi) = 0$.

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Because of points (1)–(4), $\boldsymbol{P}(X)$ turns out to be a differential operator of order one with respect to $\mathcal{D}$ and is realized as a covariant derivative in the Hermitean line bundle with compatible flat connection denoted by $(L, \langle\cdot,\cdot\rangle, \nabla)$.

A classification of unitary inequivalent Borel quantizations on the manifold Δ is given by a bijective mapping onto the set

$$\eta = \pi_1^*(\Delta) \times \mathbb{R}_2 \tag{1}$$

where $\pi_1^*(\Delta) = \mathrm{Hom}(\pi_1(\Delta), U(1))$ denotes the group of characters of the fundamental group $\pi_1(\Delta)$.

2. Application to 3-manifolds

We want to apply this classification (1) to closed orientable 3-manifolds M.

First we give a result for Δ_n and F_n constructed from any smooth manifold M with $\dim(M) \geqslant 3$. For the fundamental group of $\pi_1(\Delta_n)$ and $\pi_1(F_n)$ the following holds [5]:

$$\pi_1(\Delta_n) = S_n \otimes (\pi_1(M))^n \tag{2}$$

$$\pi_1(F_n) = (\pi_1(M))^n. \tag{3}$$

The group multiplication in $\pi_1(\Delta_n)$ has to be understood as follows, with $p_1, p_2 \in S_n$ and $l_i, k_j \in \pi_1(M), i, j \in \{1, \ldots, n\}$,

$$(p_1; k_1, \ldots, k_n)(p_2; l_1, \ldots, l_n) = (p_1 p_2; k_{p_2(1)} l_1, \ldots, k_{p_2(n)} l_n).$$

The proof of equations (2) and (3) is based on the observation that

$$i : \pi_1(F_n) \to (\pi_1(M))^n$$

is an isomorphism, if $\dim(M) \geqslant 3$ and that the sequence

$$1 \to \pi_1(F_n) \to \pi_1(\Delta_n) \to S_n \to 1$$

is splitting, if $\dim(M) \geqslant 3$ (see [3]).

Closed orientable 3-manifolds M are connected to knot theory. A refinement of Alexander's theorem [1] is given by Hilden [4] and Montesinos [7].

Theorem 2.1. (Hilden–Montesinos.) Every closed orientable 3-manifold M is an irregular three-fold branched covering of S^3. The branching set can be chosen as a knot K. M is denoted by $M = (S^3 - K)\tilde{}$.

In general the three-fold irregular branched covering of a knot complement $S^3 - K$ is not unique.

Combining the classification (1) with equation (2) and (3) and theorem (2.1) we find a *specific* classification for quantizations of particles on 3-manifolds.

Lemma 2.1. If n particles are localized on a three-dimensional closed orientable manifold M, the set of equivalence classes of quantum Borel kinematics can bijectively be mapped onto

$$\eta = (S_n \otimes (\pi_1((S^3 - K)\tilde{}))^n)^* \times \mathbb{R} \tag{4}$$

if the particles are identical and onto

$$\eta = ((\pi_1((S^3 - K)\tilde{}))^n)^* \times \mathbb{R} \tag{5}$$

if the particles are distinguishable. K is a matching knot for M.

3. Examples

3.1. Trefoil knot

Consider two (identical) particles on those three-dimensional manifolds, which arise as three-fold branched coverings of a trefoil knot complement. There are two different three-fold branched coverings over the trefoil [9]. We calculate for both of them π_1 and the unitary one-dimensional representations π_1^* [3].

The first three-fold covering is irregular and is homeomorphic to the sphere S^3. $\pi_1(S^3) = \{e\}$ is trivial. From the classification and equation (2) we obtain the usual symmetric representation of S_2 corresponding to fermions and the antisymmetric representation which corresponds to Bosons, i.e. $\eta = \{-1, +1\} \times \mathbb{R}$.

The second one is regular and denoted by $(S^3 - \text{trefoil})\tilde{}$. One obtains for $\pi_1(S^3 - \text{trefoil})\tilde{} = \langle j, k | j^4 = 1, k^2 = j^2, kj = j^3 k\rangle = Q$ the quaternion group [9] with eight elements, i.e. $|Q| = 8$. The commutator subgroup is $[Q, Q] = \{1, j^2\}$. The number of one-dimensional unitary representations U_i; $i \in \{1, \ldots, 4\}$ is $|Q|/|[Q, Q]| = 4$. They are listed in table 1.

Table 1. Unitary one-dimensional representations of the quaternion group.

	$U_1(\cdot)$	$U_2(\cdot)$	$U_3(\cdot)$	$U_4(\cdot)$
1	1	1	1	1
j	1	1	−1	−1
j^2	1	1	1	1
j^3	1	1	−1	−1
k	1	−1	1	−1
jk	1	−1	−1	1
j^2k	1	−1	1	−1
j^3k	1	−1	−1	1

For the set η of non-equivalent quantizations we obtain from (5) for two distinguishable particles $(|Q|/|[Q, Q]|)^2 = 16$ one-dimensional unitary representations corresponding to the products $U_i U_j : Q \times Q \to U(1)$, $i, j \in \{1, \ldots, 4\}$:

$$\eta = \{U_i U_j\} \times \mathbb{R} \qquad i, j \in \{1, \ldots, 4\}.$$

For two identical particles we get from (4)

$$\frac{|S_2 \circledS (Q \times Q)|}{|[S_2 \circledS (Q \times Q), S_2 \circledS (Q \times Q)]|} = \frac{128}{16} = 8$$

elements in π_1^*. If d_1 denotes the symmetric representation of S_2 and d_2 the antisymmetric one, the corresponding eight one-dimensional representations $U_{ij} : S_2 \circledS (Q \times Q) \to U(1)$, $i \in \{1, 2\}$, $j \in \{1, 2, 3, 4\}$, are given by $U_{ij}(p; l_1, l_2) = d_i(p) U_j(l_1) U_j(l_2)$.

$$\eta = \{U_{ij}\} \times \mathbb{R} \qquad i \in \{1, 2\}, \;\; j \in \{1, \ldots, 4\}.$$

3.2. Lens spaces

A geometrical description of lens spaces $L(p, q)$ is given in [9]. Their fundamental groups are $\pi_1(L(p, q)) = \mathbb{Z}_p$. The $|\mathbb{Z}_p| = p$ one-dimensional unitary representations are given by the p roots of unity. For two distinguishable particles, localized on $L(p, q)$, we obtain from (5) $|\mathbb{Z}_p \times \mathbb{Z}_p| = p^2$ inequivalent quantizations with $\eta = \{\sqrt[p]{1}\} \times \{\sqrt[p]{1}\} \times \mathbb{R}$.

For two identical particles on $L(p, q)$ an evaluation of equation (4) gives

$$\frac{|S_2 \otimes (\mathbb{Z}_p)^2|}{|[S_2 \otimes (\mathbb{Z}_p)^2, S_2 \otimes (\mathbb{Z}_p)^2]|} = \frac{2p^2}{p} = 2p$$

inequivalent quantizations, corresponding to two one-dimensional representations of S_2 and p representations of $\mathbb{Z}_p$. This gives $\eta = \{-1, +1\} \times \{\sqrt[p]{1}\} \times \mathbb{R}$.

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References

[1] Alexander J W 1920 *Bull. Am. Math. Soc.* **26** 370
[2] Angermann B, Doebner H D and Tolar J 1984 *Lecture Notes in Math.* **1037** 171
[3] Groth W 1996 *Dissertation* TU Clausthal
[4] Hilden H M 1976 *Am. J. Math.* **98** 989
[5] Imbo T D, Imbo C S and Sudershan E C G 1990 *Phys. Lett.* **234B** 1–2
Imbo T D, Imbo C S and Sudershan E C G 1990 *Phys. Lett.* **234B** 103
[6] Kostant B 1970 *Lecture Notes in Math.* **170** 87
[7] Montesinos J 1974 *Bull Am. Math Soc.* **80** 845
[8] Mueller U A and Doebner H-D 1993 *J. Phys. A: Math. Gen.* **26** 719
[9] Rolfsen D 1976 *Math Lecture Series* vol 7 (Publish or Perish)